Proof-Relevant Partial Equivalence Relations

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What's this talk about?

A partial equivalence relation (PER) is an homogeneous binary relation that is symmetric and transitive.

PERs are important in semantics of type theory and programming languages, higher-order computability and more.

To build a PER model, one starts with some realizers. Types are interpreted as PERs over realizers. When xRy we think of x and y as implementing the same program of type R.

Inspired by the homotopy interpretation of ITT, we will describe categories of proof-relevant PERs.

Realizers

Our realizers come from categorical models of the untyped λ-calculus.

A 1-categorical model of the untyped λ-calculus is a cartesian closed category $\mathcal{C}$ with a reflexive object $U \in \mathcal{C}$.

$$U^U \xrightarrow{\mathsf{lam}} U \xrightarrow{\mathsf{app}} U^U \quad = \quad \mathsf{id}$$

$$\llbracket x_1, \ldots, x_n \vdash \lambda y.t \rrbracket \coloneqq U^n \xrightarrow{\lambda \llbracket x_1, \ldots, x_n, y \vdash t \rrbracket} U^U \xrightarrow{\mathsf{lam}} U$$

$$\llbracket x_1, \ldots, x_n \vdash tu \rrbracket \coloneqq U^n \xrightarrow{\langle \mathsf{app} \circ \llbracket x_1, \ldots, x_n \vdash t \rrbracket, \llbracket x_1, \ldots, x_n \vdash u \rrbracket \rangle} U^U \times U \xrightarrow{\mathsf{eval}} U$$

Realizers

$\mathcal{C}(1, U)$ is our set of realizers.

We can apply one realizer to another:

$$(-) \cdot (-) : \mathcal{C}(1, U) \times \mathcal{C}(1, U) \to \mathcal{C}(1, U)$$
$$t \cdot u = \mathsf{eval} \circ \langle \mathsf{app} \circ t, u \rangle$$

Using the λ-calculus as an internal language for $(\mathcal{C}, U)$, we can write:

$$t \cdot u \coloneqq tu$$

Some handy realizers:

$$\mathsf{comp} \coloneqq \lambda e_1 e_2 x. e_2 (e_1 x)$$
$$\mathsf{id} \coloneqq \lambda x. x$$

The category of PERs

Objects: PERs over $\mathcal{C}(1, U)$.

A morphism $R \to S$ is a function

$$f : {}^{\mathcal{C}(1,U)}\!\big/_{R} \to {}^{\mathcal{C}(1,U)}\!\big/_{S}$$

between quotients such that

$$\exists e \in \mathcal{C}(1, U).\ \forall t R t.\ f[t] = [e \cdot t]$$

We write $e \Vdash f$ and say that e tracks (or realizes, or implements) f.

If $e_1 \vdash f$ and $e_2 \Vdash g$ then comp $e_1 e_2 \Vdash gf$. id tacks identities.

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But wait! How do we quotient by a PER?

Quotients of PERs: the standard way

Let R be a PER over the set X.

Define $\mathsf{Dom}(R) \coloneqq \{x \in X \mid xRx\}$.

The quotient ${}^{X}\!/_{R} \coloneqq {}^{\mathsf{Dom}(R)}\!/_{R}$.

Quotients of PERs: the interesting way

A semicategory is a category without identities.

The forgetful functor from categories to semicategories has both a left and a right adjoint. The right adjoint is the Karoubi envelope construction K.

Given a semicategory $\mathcal{S}$, the category $K\mathcal{S}$ has:

- objects: (A, a), where a is an idempotent on A;
- morphisms $(A, a) \to (B, b)$: maps $f : A \to B$ such that $fa = f$ and $bf = f$;
- composition: inherited from $\mathcal{S}$;
- the identity on (A, a) is a.

A semifunctor from a category to a semicategory takes identities to idempotents.

Quotients of PERs: the interesting way

A (partial) equivalence relation R over X can be thought of as a (semi)category (X, R) (in $\mathbf{Set}$).

$$R \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} X$$

There is a map $x \to y$ whenever xRy.

As categories: $KR \cong (\mathsf{Dom}(R), R)$.

The quotient of an equivalence relation as above is its coequalizer in $\mathbf{Set}$.

Proof-relevant relations

A(nother) perspective on the relation R is that it is a function

$$R : X \times X \to 2$$

This function tells us **when** x is related to y.

A **proof-relevant** relation R on a category $\mathcal{X}$ is a **functor**

$$R : \mathcal{X}^{\mathsf{op}} \times \mathcal{X} \to \mathbf{Set}$$

The functor gives us a set of "proofs" that x is related to y (there may be no such proofs), as well as a way to transport proofs along morphisms in $\mathcal{X}$.

Proof-relevant relations

A functor $R : \mathcal{X}^{\mathsf{op}} \times \mathcal{X} \to \mathbf{Set}$ corresponds via the two-sided Grothendieck construction to a two-sided discrete fibration

$$\int R \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathcal{X}$$

The category $\int R$ has:

- objects: $(x, y, p \in R(x, y))$;
- morphisms $(x, y, p) \to (x', y', p')$: pairs $(f : x \to x', g : y \to y')$ such that $R(x, g)(p) = R(f, y')(p')$.

Proof-relevant relations

A **catead** is a category in $\mathbf{Cat}$, ie. a double category, such that the source-target span is a two-sided discrete fibration.

Thus a catead is a proof-relevant relation with the structure of composition and identities, corresponding to transitivity and reflexivity respectively.

These behave in $\mathbf{Cat}$ as equivalence relations do in $\mathbf{Set}$ (they are effective) [Bourke]. (There is a groupoidal version of this story that features symmetry, but it is a bit more complicated.)

In particular, we can take the **codescent object** (higher quotient) of a catead, which "coequalizes" s and t up to isomorphism. The codescent object of a catead is its horizontal category.

Proof-relevant relations

A proof-relevant *partial* equivalence relation is a **semicatead**, ie. a semicategory in $\mathbf{Cat}$ such that the source-target span is a two-sided discrete fibration.

The forgetful functor from $\mathbf{Cat}(\mathbf{Cat})$ to $\mathbf{SemiCat}(\mathbf{Cat})$ has a right adjoint $\mathbb{K}$ (the double-categorical Karoubi envelope). Given $\mathbb{S} \in \mathbf{SemiCat}(\mathbf{Cat})$ we define $\mathbb{KS} \in \mathbf{Cat}(\mathbf{Cat})$:

- The horizontal category $(\mathbb{KS})_h$ is the Karoubi envelope $K(\mathbb{S}_h)$ of the horizontal category $\mathbb{S}_h$.
- A vertical morphism is an idempotent square (wrt horizontal composition).
- A square $\alpha \to \alpha'$ is a square β satisfying $\beta\alpha = \beta$ and $\alpha'\beta = \beta$.

$\mathbb{K}$ takes semicateads to cateads.

2D realizers

A 2D model of the untyped λ-calculus is a cartesian closed bicategory $\mathfrak{C}$ with a pseudoreflexive object $U \in \mathfrak{C}$.

$$U^U \xrightarrow{\text{lam}} U \xrightarrow{\text{app}} U^U \quad \cong \quad \text{id}$$

Examples:

- generalised species of structures [Fiore, Gambino, Hyland, Winskel]
- profunctorial Scott semantics [Galal]
- categorified relational (distributors-induced) model [Olimpieri]
- categorified graph model [Kerinec, Manzonetto, Olimpieri]

We have a **category** $\mathfrak{C}(1, U)$ and an application **functor**.

The category of proof-relevant PERs

An object is a semicatead whose category of objects is $\mathfrak{C}(1, U)$.

A morphism $\mathbb{S} \to \mathbb{S}'$ is a functor

$$F : Q(\mathbb{KS}) \to Q(\mathbb{KS}')$$

between codescent objects of Karoubi envelopes such that

$$\exists e \in \mathfrak{C}(1, U).\ Fq \cong q(K(\hat{e}))$$

where $q : (\mathbb{KS})_{\mathsf{v}} \to Q(\mathbb{KS}) = (\mathbb{KS})_{\mathsf{h}}$ is the codescent functor and $\hat{e} \coloneqq e \cdot (-) : \mathbb{S}_{\mathsf{v}} \to \mathbb{S}'_{v}$.

Future work

- 2D model of System F
- Model of HoTT
- Relation to other realizability models: assemblies, realizability toposes
- Connections to proof-relevant parametricity [Ghani, Forsberg, Orsanigo]